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Ising replica magnets

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Abstract. The phase structure of systems of n replicas of Ising systems having both inter-and intra-replica exchange is investigated within mean field theory. Cooperative phases are found with inter-replica ordering both with and without intra-replica ordering. First-order and second-order transitions are found.

1. Introduction

In their seminal paper on spin glasses, Edwards and Anderson (1975, to be referred to as EA) employed the now (in)famous replication procedure in which the average free energy of a quenched spatially disordered system is investigated via the mathematical identity

$$\overline{\ln Z} = \lim_{n\to 0} \frac{1}{n}(\overline{Z^n} - 1), \tag{1.1}$$

where the bar denotes an average over the spatial disorder. Z^n is identified as the partition function of n replicas of the original system and $\overline{Z^n}$ is interpreted as the partition function of an effective pure system. Unfortunately the practical evaluation of $\overline{Z^n}$ and application of the limit has involved approximations and/or assumptions of less rigour then (1.1): even superficially 'exact' procedures (Sherrington and Kirkpatrick 1975) can lead to unphysical results when applied to systems with quenched spatially random distributions of frustrated interactions. (A frustrated system is one in which no state can be found which satisfies all the exchange interactions (Toulouse 1977).) Much effort has been put into trying to find a useful valid application of the $n \to 0$ replica trick for such systems, but so far without success.

The procedure of averaging Z^n over the spatial disorder yields effective Hamiltonians which are lattice-translationally invariant but which have more complicated interactions than those of the original (disordered) system. Conventionally the spin-glass literature has (naturally) emphasised the limitingly small n behaviour. In this paper, however, we consider the simplest classes of Hamiltonians and effective Hamiltonians suggested by the $\overline{Z^n}$ procedure but concentrate on finite integral values of n. Suzuki (1977) has also considered the finite-replica problem but appears to have ignored the possibility of first-order transitions, which we show to be important.

The class of systems we study here are those described by Hamiltonians of the form

$$H = -\sum_{(ij)} \sum_{\alpha} J_{ij}^{(1)} S_i^{\alpha} S_j^{\alpha} - \sum_{(ij)} \sum_{(\alpha\beta)} J_{ij}^{(2)} S_i^{\alpha} S_j^{\alpha} S_i^{\beta} S_j^{\beta} - h \sum_i \sum_{\alpha} S_i^{\alpha}, \tag{1.2}$$

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where the S_i^α are Ising-like spins taking the values ± 1, the subscripts refer to spatial location and run over all N sites of a lattice, the α, β label 'replicas' and may take n values, and the symbols (i, j) and (α, β) refer to pairs of *non-identical* indices. For a physical system n must be integral and positive. For $n = 1$ the $J^{(2)}$ interaction vanishes, so interest starts at $n = 2$. The thermodynamic limit is $N \to \infty$ and does not affect n†. Extensions of (1.2) to general m-dimensional spins (vector or Potts) are readily envisaged but will not be discussed here. For simplicity we restrict our discussion to cases in which the Fourier-transformed exchange interactions $J^{(1)}(\boldsymbol{k})$, $J^{(2)}(\boldsymbol{k})$ have their maxima at $k = 0$; any magnetic order is then translationally invariant.

Equation (1.2) can be considered as a generalisation of the Ashkin–Teller model (Ashkin and Teller 1943). In its general form it does not seem to have been studied systematically. Certain special cases related to known models can, however, be identified:

(i) if $J^{(2)} = 0$, the model reduces to n uncoupled Ising models.

(ii) in the limit $J^{(2)}$ infinitely larger than $J^{(1)}$, h, or T, there are projected out as relevant only the states in which the $\boldsymbol{S}_i = (S_i^1, \ldots, S_i^n)$ are parallel or antiparallel from site to site; that is $\{S_i\} = \pm\{S_j\}$. If h is zero all such states are of importance; if h is non-zero the only ones relevant in the thermodynamic limit have all the replica components on any site identical. In either case the relevant part of the free energy (apart from uninteresting constant shifts) is

$$F = -kT \ln Z^*,$$

where Z^* is the partition function of a simple $(S^z = \pm 1)$ Ising model with effective exchange $nJ^{(1)}$ and effective magnetic field nh.

(iii) for $n = 2$, if $J^{(2)} = 2J^{(1)}$ and $h = 0$, the energy associated with any pair of spins (i, j) is the same for all combinations of $\{S_i^\alpha\}$, $\{S_j^\beta\}$ which are not identical on each of the sites, and is different from that with identical spins on each site (which, further, is itself independent of the precise set). The model thus reduces in this case to a four-state Potts model.

(iv) if $n = 2$, $d = 2$ (d is the spatial dimension), one obtains a standard Ashkin–Teller model; for a recent discussion see Knops (1975).

Equation (1.2) is also reminiscent of the discrete spin cubic (DSC) model (Kim *et al* 1975, Kim and Levy 1975, Kim *et al* 1976, Aharnoy 1977). For the case of both dipolar and quadrupolar exchange the DSC Hamiltonian is identical with (1.2) but differs in its allowed spin space. The space for the DSC model is that in which, for each i, any one S_i^α can take the values ± 1, but the rest are zero; it has only $2n$ elements per site as compared with our 2^n. The models are equivalent only for $n = 2$.

The model of (1.1) as it stands does not result from an Edwards–Anderson treatment of a Hamiltonian spin glass, but there can arise the related effective 'Hamiltonian' with

$$J_{ij}^{(2)} \to J^2/kT; \tag{1.3}$$

this effective Hamiltonian results from averaging Z^n for a random Ising system

$$H = -\sum_{(ij)} J_{ij} S_i S_j, \tag{1.4}$$

† One dubious procedure in the spin-glass analyses concerns interchange of the limits $n \to 0$ and $N \to \infty$. Here n is assumed fixed.

with the J_{ij} distributed randomly and independently with probability distribution

$$p(J_{ij}) = (2\pi J^2)^{-1/2} \exp[-(J_{ij} - J^{(1)})^2/2J^2] \tag{1.5}$$

(Sherrington and Kirkpatrick 1975). The EA treatment further takes the limit $n \to 0$ but we shall consider only finite n†. We shall present results based on (1.1); the corresponding results for the 'EA'-type model follow from the transformation (1.2).

In this paper we treat (1.1) only within the mean field approximation‡, since even such an approximation presents an interesting phase diagram. Mean field theory becomes exact in the limit where the spatial sums are taken over all sites with the scaling

$$J_{ij}^{(r)} = \tilde{J}^{(r)}/N, \qquad r = 1, 2. \tag{1.6}$$

2. Mean field theory

Within mean field theory one introduces the order parameters§

$$m^\alpha = \langle S_i^\alpha \rangle, \tag{2.1}$$

$$q^{(\alpha\beta)} = \langle S_i^\alpha S_i^\beta \rangle, \tag{2.2}$$

(Sherrington 1975) and employs the substitutions

$$\sum_{(ij)} J_{ij}^{(1)} S_i^\alpha S_j^\alpha \to m^\alpha \Big(\sum_j J_{ij}^{(1)} \Big) \sum_i (S_i^\alpha - m^\alpha/2), \tag{2.3}$$

$$\sum_{(ij)} J_{ij}^{(2)} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta \to q^{(\alpha\beta)} \Big(\sum_j J_{ij}^{(2)} \Big) \sum_i (S_i^\alpha S_i^\beta - q^{(\alpha\beta)}/2). \tag{2.4}$$

Using the shorthand

$$\sum_j J_{ij}^{(r)} = J^{(r)}, \qquad r = 1, 2, \tag{2.5}$$

the free energy per spin becomes

$$f \equiv \frac{F}{N} = \sum_\alpha J^{(1)} (m^\alpha)^2/2 + \sum_{(\alpha\beta)} J^{(2)} (q^{(\alpha\beta)})^2/2 - kT \ln \mathrm{Tr} \exp\Big[\sum_\alpha (J^{(1)} m^\alpha + h) S^\alpha/kT + (J^{(2)}/kT) \sum_{(\alpha\beta)} q^{(\alpha\beta)} S^\alpha S^\beta \Big] \tag{2.6}$$

where the trace is single-site. Minimisation of f with respect to m^α and $q^{(\alpha\beta)}$ yields their self-consistency equations

$$m^\alpha = \mathrm{Tr}\, S^\alpha \Phi / \mathrm{Tr}\, \Phi, \tag{2.7}$$

$$q^{(\alpha\beta)} = \mathrm{Tr}\, S^\alpha S^\beta \Phi / \mathrm{Tr}\, \Phi, \tag{2.8}$$

where Φ is the exponential in (2.6).

† See the previous footnote.

‡ The model is in fact exactly soluble in one dimension (using transfer matrix techniques) but the results are not typical of higher dimensions. For $d = 2$ or 3, real-space renormalisation-group methods can also be employed.

§ Were the maxima of $J^{(1)}(\boldsymbol{k})$ and $J^{(2)}(\boldsymbol{k})$ at non-zero k-values we should need to employ corresponding staggered order parameters.

For general n the number of possible solution symmetries is very large, but it may be shown (van Hemmen and Palmer 1979) that the solutions of (2.7) and (2.8) which are absolute minima of (2.6) have the properties

$$|q^{(\alpha\beta)}| = q, \tag{2.9}$$

$$m^\alpha = m, \tag{2.10}$$

with all the $q^{(\alpha\beta)}$ real and such that for $n \geqslant 3$ for any set (α, β, γ) the product $q^{(\alpha\beta)}q^{(\beta\gamma)}q^{(\gamma\alpha)}$ is positive. Without loss of generality we consider the symmetric solution

$$q^{(\alpha\beta)} = q; \tag{2.11}$$

the other possibilities follow by simple transformation. The identity

$$\exp\Big(\eta \sum_{(\alpha\beta)} S^\alpha S^\beta\Big) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dz \exp\Big(-z^2/2 + \eta^{1/2} z \sum_\alpha S^\alpha - n\eta/2\Big) \tag{2.12}$$

yields the free energy and correlation functions in the compact forms

$$f = nJ^{(1)}m^2/2 + n(n-1)J^{(2)}q^2/4 + nJ^{(2)}q/2$$
$$- kT \ln((2\pi)^{-1/2} \int_{-\infty}^{\infty} dz \exp(-z^2/2)(2\cosh \Xi)^n, \tag{2.13}$$

$$\langle S^\alpha \ldots S^\gamma \rangle_{\substack{r \text{ indices} \\ \text{all different}}} = \frac{\int_{-\infty}^{\infty} dz \exp(-z^2/2) \tanh^r \Xi \cosh^n \Xi}{\int_{-\infty}^{\infty} dz \exp(-z^2/2) \cosh^n \Xi}, \tag{2.14}$$

where

$$\Xi = (J^{(1)}m + h)/kT + (J^{(2)}q/kT)^{1/2} z. \tag{2.15}$$

m and q are given self-consistently by those solutions of (2.14), with $r = 1, 2$ respectively, which minimise f.

We shall refer to the possible phases as follows:

$m = 0, q = 0$; paramagnet; $m = 0, q \neq 0$; replica magnet;

$m \neq 0, q \neq 0$; ferromagnet.

3. Single exchange

In this section we consider the simplified case $J^{(1)} = 0$, $h = 0$.

It is convenient to rewrite the self-consistency equation for q in terms of

$$x = J^{(2)}q/kT;$$

the self-consistency condition is then

$$\frac{kTx}{J^{(2)}} = g(x) \equiv \frac{\int_{-\infty}^{\infty} dz \exp(-z^2/2) \tanh^2(x^{1/2}z) \cosh^n(x^{1/2}z)}{\int_{-\infty}^{\infty} dz \exp(-z^2/2) \cosh^n(x^{1/2}z)}. \tag{3.1}$$

For $n = 2$ equation (3.2) yields a second-order transition, reducing to the self-consistency condition of a pure Ising model,

$$(kT/J^{(2)})x = \tanh x, \tag{3.2}$$

as indeed is apparent directly from (1.1). For $n > 2$ the transition is of first order. This is immediately apparent from the small-x expansion of $g(x)$,

$$g(x) = x + (n-2)x^2 + \cdots. \tag{3.3}$$

We further see immediately that in the low-temperature phase the solution of (3.1) may have q of either sign for $n = 2$ but has positive q for $n > 2$. Table 1 gives critical temperatures and order parameters for various n. The explicit calculation for $n \to \infty$ is given in the Appendix. Figure 1 shows $q(T)$, which is finite for $T < T_c$, zero for $T > T_c$.

The zero-field susceptibility is simply expressed in terms of $q(T)$:

$$\chi(J^{(1)} = 0, h = 0) = (kT)^{-1}[1 + (n-1)q(T)]. \tag{3.4}$$

Table 1. Transition temperature and order parameters for $J^{(1)} = h = 0$. Temperature is measured in units of $J^{(2)}/k$. p is $\langle S^\alpha S^\beta S^\gamma S^\delta \rangle$ with all indices different.

n	T_c	q_c	p_c
2	1	0	0
3	1·2137	0·6667	0
4	1·5318	0·8202	0·7322
5	1·8731	0·8839	0·8162
$\sim\infty$	0·363n	0·981	0·962

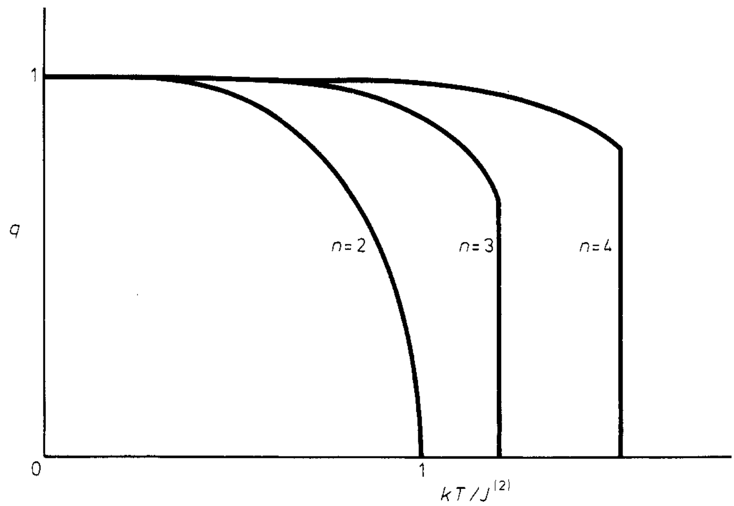

Figure 1. $q(T)$ for $n = 2, 3, 4$.

4. Dual exchange

When both $J^{(1)}$ and $J^{(2)}$ are non-zero, the self-consistent solution for m and q is more complicated. Let us concentrate on $h = 0$. The phase diagrams are then as illustrated in figure 2; those for $n > 2$ are given only for the lower end of the $T, J^{(1)}$ range, but for larger $T, J^{(1)}$ have the same qualitative form as for $n = 2$. For all values of n there are both first- and second-order regimes. The transitions between paramagnetism and

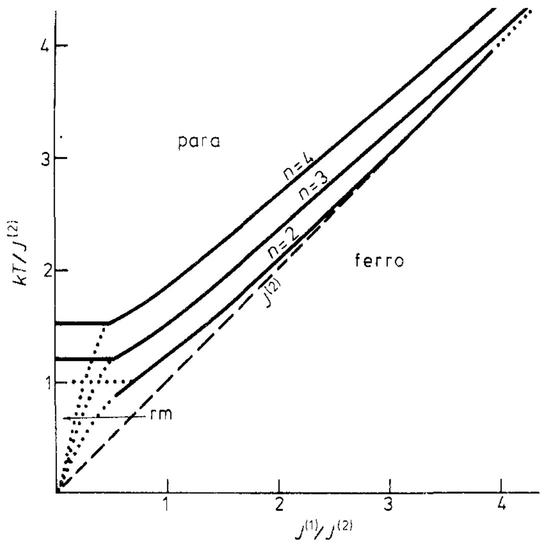

Figure 2. Phase diagram for $n = 2, 3, 4$. Full lines indicate first-order transitions. Dotted lines indicate second-order lines. The broken line indicates the ferromagnetic transition line in the absence of inter-replica exchange. For $n > 2$ only the lower T, $J^{(1)}$ region of the phase diagram is shown. In all cases the paramagnetic–ferromagnetic transition approaches the $T = J^{(1)}$ line with a crossover to second-order behaviour at $T = [3(n-1)+1]J^{(2)}/k$. For $n = 2$ and $n = 3$ the replica-magnet to ferromagnet transition has a crossover from second- to first-order at $kT/J^{(2)} = 0{\cdot}8537$, $1{\cdot}1893$ respectively.

replica-magnet ordering are as obtained in the last section; the origins of the transition orders of paramagnetic–ferromagnetic and replica-magnet to ferromagnet transitions can be seen from an expansion of the self-consistency conditions in terms of power series in m.

With the shorthand notations

$$x = J^{(2)}q/kT, \tag{4.1a}$$

$$y = J^{(1)}m/kT, \tag{4.1b}$$

$$\Xi = x^{1/2}z + y, \tag{4.2}$$

the self-consistency equations are

$$\left(\frac{kT}{J^{(1)}}\right)y = \frac{\mathrm{Tr}_n\, S^1 \exp(y\,\Sigma_\alpha S^\alpha + x\,\Sigma_{(\alpha\beta)} S^\alpha S^\beta)}{\mathrm{Tr}_n \exp(y\,\Sigma_\alpha S^\alpha + x\,\Sigma_{(\alpha\beta)} S^\alpha S^\beta)} \tag{4.3a}$$

$$= \frac{\int_{-\infty}^{\infty} \mathrm{d}z\, \exp(-z^2/2) \tanh \Xi \cosh^n \Xi}{\int_{-\infty}^{\infty} \mathrm{d}z\, \exp(-z^2/2) \cosh^n \Xi}, \tag{4.3b}$$

$$\left(\frac{kT}{J^{(2)}}\right)x = \frac{\mathrm{Tr}_n\, S^1 S^2 \exp(y\,\Sigma_\alpha S^\alpha + x\,\Sigma_{(\alpha\beta)} S^\alpha S^\beta)}{\mathrm{Tr}_n \exp(y\,\Sigma_\alpha S^\alpha + x\,\Sigma_{(\alpha\beta)} S^\alpha S^\beta)} \tag{4.4a}$$

$$= \frac{\int_{-\infty}^{\infty} \mathrm{d}z\, \exp(-z^2/2) \tanh^2 \Xi \cosh^n \Xi}{\int_{-\infty}^{\infty} \mathrm{d}z\, \exp(-z^2/2) \cosh^n \Xi}. \tag{4.4b}$$

Expanding (4.3) and (4.4) in powers of y gives

$$\left(\frac{kT}{J^{(1)}}\right)y = y[1+(n-1)q_0]+y^3[-\tfrac{1}{3}-\tfrac{4}{3}(n-1)q_0-\tfrac{1}{2}n(n-1)^2q_0$$
$$+\tfrac{1}{6}(n-1)(n-2)(n-3)p_0]+\mathrm{O}(y^5), \tag{4.5}$$

$$(kT/J^{(2)})x = q_0+y^2[1+2(n-2)q_0-\tfrac{1}{2}n(n-1)q_0^2+\tfrac{1}{2}(n-2)(n-3)p_0]+\mathrm{O}(y^4), \tag{4.6}$$

where

$$q_0=\frac{\mathrm{Tr}_n\, S^1S^2\exp(x\,\Sigma_{(\alpha\beta)}\,S^\alpha S^\beta)}{\mathrm{Tr}_n\,\exp(x\,\Sigma_{(\alpha\beta)}\,S^\alpha S^\beta)} \tag{4.7a}$$

$$=\frac{\int_{-\infty}^{\infty}\mathrm{d}z\,\exp(-z^2/2)\tanh^2(x^{1/2}z)\cosh^n(x^{1/2}z)}{\int_{-\infty}^{\infty}\mathrm{d}z\,\exp(-z^2/2)\cosh^n(x^{1/2}z)}, \tag{4.7b}$$

$$p_0=\frac{\mathrm{Tr}_n\, S^1S^2S^3S^4\exp(x\,\Sigma_{(\alpha\beta)}\,S^\alpha S^\beta)}{\mathrm{Tr}_n\,\exp(x\,\Sigma_{(\alpha\beta)}\,S^\alpha S^\beta)} \tag{4.8a}$$

$$=\frac{\int_{-\infty}^{\infty}\mathrm{d}z\,\exp(-z^2/2)\tanh^4(x^{1/2}z)\cosh^n(x^{1/2}z)}{\int_{-\infty}^{\infty}\mathrm{d}z\,\exp(-z^2/2)\cosh^n(x^{1/2}z)}. \tag{4.8b}$$

As is clear from (4.5) and (4.6) p is only relevant for $n\geqslant 4$. Note that q_0 and p_0 are implicit functions of y through (4.6). Solving for q_0 to order y^2, (4.5) can be expressed in explicit orders in y as

$$(kT/J^{(1)})y = y[1+(n-1)q_{00}]$$
$$+y^3\Big(-\tfrac{1}{3}-\tfrac{4}{3}(n-1)q_{00}-\tfrac{1}{2}n(n-1)^2q_{00}^2+\tfrac{1}{6}(n-1)(n-2)(n-3)p_0$$
$$+\frac{(n-1)[1+2(n-2)q_{00}-\tfrac{1}{2}n(n-1)q_{00}^2+\tfrac{1}{2}(n-2)(n-3)p_{00}]}{\{(kT/J^{(2)})[1+2(n-2)q_{00}-\tfrac{1}{2}n(n-1)q_{00}^2+\tfrac{1}{2}(n-2)(n-3)p_{00}]^{-1}-1\}}\Big)$$
$$+\mathrm{O}(y^3), \tag{4.9}$$

where q_{00}, p_{00} are given by (4.7) and (4.8), with x determined by (4.6) with $y=0$; that is by the values resulting from the treatment of § 3. The order of the transition to ferromagnetism is then determined by the sign of the coefficient of y^3 in (4.9); if negative the transition is second order, if positive, it is first order.

From (4.9) we see immediately that if we are in a region of $(J^{(2)}, T)$ space for which q_{00} (and thus p_{00}) is zero the coefficient of y^3 in (4.9) becomes

$$-\tfrac{1}{3}+(n-1)(kT/J^{(2)}-1)^{-1}, \tag{4.10}$$

so that the transition is second order if $T>T_c^{(2)}$, first order if $T<T_c^{(2)}$, where

$$T_c^{(2)}=(3n-2)J^{(2)}/k. \tag{4.11}$$

For $T>T_c^{(2)}$ the transition temperature is given by

$$T_c=J^{(1)}/k. \tag{4.12}$$

As T tends to zero, p_{00} and q_{00} both tend to unity and the coefficient of y^3 thus tends to $-n^3/3$ which is always negative (for the n of interest here). It thus follows that the low-temperature transition from replica-magnet to ferromagnet is second order with

$$T_c = (J^{(1)}/k)[1+(n-1)q_{00}(T_c)]. \tag{4.13}$$

Note that as $J^{(2)} \to \infty$, $q_{00}(T_c) \to 1$ so that $T_c \to nJ^{(1)}/k$ which is the mean-field transition temperature of an Ising model with exchange $nJ^{(1)}$. This is as expected on the basis of special case (ii) of § 1. For $n=2$ and $n=3$ the replica-magnet to ferromagnet transitions become first order at $kT/J^{(2)} = 0{\cdot}8537$, $1{\cdot}1893$ respectively. For $n \geq 4$ the transition is everywhere second order.

In the regions of parameter space for which the spontaneous magnetisation is zero the zero-field susceptibility is given by

$$\chi(m=0, n=0) = (\chi_0^{-1} - J^{(1)})^{-1}, \tag{4.14}$$

where χ_0 is the susceptibility of the system with $J^{(1)} = 0$, given by equation (3.4).

5. Effect of a magnetic field on the replica-magnet transition

The zero-field susceptibility has already been given above in equations (3.4) and (4.13). In the $n \to 0$ limit it has been shown (Sherrington and Kirkpatrick 1975, Fischer 1976) that a finite field removes the 'spin-glass' phase transition, replacing the cusp in the susceptibility by a smooth peak. We here consider the corresponding question for the finite-n replica-magnet transition. We take $J^{(1)}$ to be zero.

For $n \geq 3$ the zero-field transition is first order. This character is maintained also in a small field, albeit that q is field-dependent both above and below the transition. For $n=2$ the zero-field transition is second order and can be seen to be smoothed out by a small field. Expanding the self-consistency equation for q in powers of h, one obtains for $n=2$

$$q \equiv Tx/J^{(2)} = \tanh x + (\beta h)^2(1-\tanh^2 x) + O(h^4), \tag{5.1}$$

so that q varies continuously as a function of temperature for h finite; cf the related expression for a pure ferromagnet,

$$Tx/J = \tanh x + \beta h + O(h^2). \tag{5.2}$$

6. Stability of the solutions

Our choice of all m^α equal and all $q^{(\alpha\beta)}$ equal is justified by a proof of Lieb (see van Hemmen and Palmer 1979), but, in view of a recent criticism by de Almeida and Thouless (1978) of this choice when made by Sherrington and Kirkpatrick (1975), it seems appropriate to demonstrate the stability of our choice also by the method of de Almeida and Thouless. We give the analysis only for the case $J^{(1)} = h = 0$. A similar analysis proves the stability in the general case.

The de Almeida–Thouless procedure consists of expanding f to quadratic order in $(m^\alpha - m)$, $(q^{(\alpha\beta)} - q)$ and examining the eigenvalues of the fluctuations. Stability requires that they be positive definite. The expansion is given by de Almeida and Thouless (1978) and here we present only the results.

For $J^{(1)} = h = 0$ the eigenvalues associated with $(q^{(\alpha\beta)} - q)$ for integers $n \geqslant 3$ are

$$\lambda_1 = 1 - (J^{(2)}/kT)[1 + 2(n-2)q_0 - \tfrac{1}{2}n(n-1)q^2 + \tfrac{1}{2}(n-2)(n-3)p], \tag{6.1a}$$

$$\lambda_2 = 1 - (J^{(2)}/kT)[1 + (n-4)q - (n-3)p], \tag{6.1b}$$

$$\lambda_3 = 1 - (J^{(2)}/kT)(1 - 2q + p), \tag{6.1c}$$

where

$$q = \langle S^\alpha S^\beta \rangle, \qquad \alpha \neq \beta, \tag{6.2}$$

$$p = \langle S^\alpha S^\beta S^\gamma S^\delta \rangle, \qquad \text{all indices different}, \tag{6.3}$$

with the correlation functions being taken in the uniform mean-field state; q and p are given by (4.7) and (4.8) with x chosen self-consistently. The degeneracies of the above modes are

$$g_1 = 1, \qquad g_2 = (n-1), \qquad g_3 = n(n-3)/2. \tag{6.4}$$

For $n = 2$ there is only one eigenfunction with eigenvalue λ_1. Eigenmodes λ_2, λ_3 are spurious, albeit with 'cancelling' degeneracies 1, -1!

For $T > T_c$, q and p are both zero so that the above eigenvalues all become $(1 - J^{(2)}/kT)$, which is positive for all $T > T_c$ since $T_c \geqslant J^{(2)}$ (see table 1). For $n = 2$ the eigenvalue becomes zero exactly at $T = T_c$ as expected for a second-order transition.

For $T < T_c$, substitution of p, q values readily shows all the eigenvalues to be positive. λ_1 is simply expressed in terms of $g(x)$ (see § 3) as

$$\lambda_1 = 1 - (J^{(2)}/kT)(\partial g(x)/\partial x). \tag{6.5}$$

It is clear from the self-consistency condition (3.1) that this is always positive, becoming zero at $T = T_c$ only for $n = 2$ where the transition is second order and the softening of the above mode is its signal.

7. Conclusion

In this paper we have considered a class of Hamiltonians which are suggested by recent analyses of spin-glass systems. They correspond to sets of Ising replicas with both intra- and inter-replica exchange. Spin-glass analysis is concerned with a limiting behaviour as the number of replicas, n, is let tend to zero. Here analysis has been of the finite replica case. The discussion has been limited to mean field theory but extensions could be made in standard ways, as could generalisation to higher-dimensional spins.

We have shown that these systems can exhibit two types of cooperatively ordered phases: (i) with both inter- and intra-replica spin alignment, (ii) with inter- but not intra-replica alignment—we call these ferromagnetic and replica-magnetic ordering respectively. With only inter-replica exchange one obtains only replica-magnetic ordering with the transition to paramagnetism of second order for $n = 2$, first order for $n > 2$. When intra-replica exchange is also present, ferromagnetic order is possible in certain regions of parameter space. For all n the transition to ferromagnetism has both second-order and first-order sections. Analytic continuation of the phase diagrams towards $n \to 0$ would thus be non-trivial; on the other hand the self-consistency equations for the order parameters are expressible in a form which is analytically continuable and indeed has been the basis for the spin-glass analyses (Edwards and

Anderson 1975, Sherrington and Kirkpatrick 1975), although the validity of such a procedure is questionable (Sherrington and Kirkpatrick 1975, de Almeida and Thouless 1978).

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Appendix

Large n limit

It is well known that the thermodynamics of certain classical m-vector ferromagnetic models can be solved exactly in the limit $m \to \infty$ (Berlin and Kac 1952, Stanley 1968). In the present problem it is tempting to look for a corresponding solution in the limit $n \to \infty$. It is clear that this limit can be relevant only if $J^{(2)}$ is important (for $J^{(2)}$ zero the problem reduces to n independent Ising models). Unfortunately, even for the $J^{(2)}$ interaction only, this problem does not seem to be exactly soluble, but one can find a formal solution in terms of (in general) non-exactly soluble Ising models—of course in the special cases where the Ising models are soluble then so is the whole problem. On the other hand we are able to solve exactly the mean field equation (2.13).

Let us start by considering the solution of the mean field equation (2.13) with $J^{(1)}$, $h=0$, and look only for T_c, q_c for large n. If we choose units $J^{(2)}=n^{-1}$, the resulting T_c, q_c are n-independent to dominant order for $n\to\infty$. The transition is clearly first order. T_c, q_c are given by: (i) equating to zero the large-n limit of the difference between the free energy evaluated from (2.13) with $q=q_c$ and that with $q=0$, (ii) requiring self-consistency for q_c as evaluated from (2.14). The free energy condition gives

$$\frac{n}{4}(q_c)^2 - kT_c \ln\left((2\pi n^{-1})^{1/2}\int_{-\infty}^{\infty} d\tilde{z}\, \exp\left[-n\left\{\frac{\tilde{z}^2}{2} - \ln\cosh\left[\tilde{z}\left(\frac{q_c}{kT_c}\right)^{1/2}\right]\right\}\right]\right) = 0. \quad (A1)$$

For large n the integral is dominated by the maxima of the exponent, giving

$$(q_c^2/4) + (q_c\mu^2/2) - kT_c \ln\cosh(q_c\mu/kT_c), \quad (A2)$$

where μ is a non-trivial solution of

$$\mu = \tanh(q_c\mu/kT_c). \quad (A3)$$

μ is thus the mean-field order parameter of an effective Ising model of exchange strength q_c. q_c itself is similarly determined by extremum-dominated integrals and its self-consistency equation yields (using (A3))

$$q_c = \mu^2, \quad (A4)$$

so that finally we have the critical conditions

$$\mu = \tanh(\mu^3/kT_c), \quad (A5)$$

$$3\mu^4/4 = kT_c \ln\cosh(\mu^3/kT_c). \quad (A6)$$

Solving, we obtain

$$kT_c = 0{\cdot}363, \qquad q_c = 0{\cdot}981.$$

Let us turn now to the investigation of the model in the large-n limit without approximation. We again start with the case $J^{(1)} = h = 0$. Apart from an uninteresting constant, the free energy is

$$F = -kT\,\mathrm{Tr}_n \exp\left[\sum_{(ij)} \left(\frac{\tilde{J}_{ij}}{2kT}\right)\left(\sum_\alpha S_i^\alpha S_j^\alpha\right)^2\right]$$

$$= -kT\,\mathrm{Tr}_n \int \prod_{(ij)} \left\{(2\pi)^{1/2}\,\mathrm{d}x_{ij} \exp\left[-\frac{x_{ij}^2}{2} + \left(\frac{J_{ij}}{kT}\right)^{1/2} x_{ij} \sum_\alpha S_i^\alpha S_j^\alpha\right]\right\}$$

$$= \int \left[\prod_{(ij)} (2\pi n^{-1})^{1/2}\,\mathrm{d}x_{ij}\right] \exp\left[-n\left(\sum_{(ij)} x_{ij}^2/2 - \ln Z_{\mathrm{eff}}\right)\right], \tag{A7}$$

where

$$Z_{\mathrm{eff}} = \mathrm{Tr} \exp\left[\sum_{(ij)} (\tilde{J}_{ij}/kT)^{1/2} x_{ij} S_i S_j\right], \tag{A8}$$

the last trace being over ordinary Ising spins. For $n \to \infty$, steepest-descent analysis gives the overwhelming contribution to the free energy,

$$F = n\left[\sum_{(ij)} (\lambda_{ij}^2/2) - kT \ln \tilde{Z}_{\mathrm{eff}}\right], \tag{A9}$$

where

$$\tilde{Z}_{\mathrm{eff}} = \mathrm{Tr} \exp\left(\sum_{(ij)} \tilde{J}_{ij}^{1/2} \lambda_{ij} S_i S_j / kT\right) \tag{A10}$$

and $\{\lambda_{ij}\}$ is determined by minimising F; that is

$$\lambda_{ij} = \tilde{J}_{ij}^{1/2} \frac{\mathrm{Tr}\, S_i S_j \exp(\Sigma_{(lm)} \tilde{J}_{lm}^{1/2} \lambda_{lm} S_l S_m / kT)}{\mathrm{Tr} \exp(\Sigma_{(lm)} \tilde{J}_{lm}^{1/2} \lambda_{lm} S_l S_m / kT)} \tag{A11}$$

$$\equiv \tilde{J}_{ij}^{1/2} \langle S_i S_j \rangle_{\mathrm{eff}}. \tag{A12}$$

Thus a knowledge of the properties of an Ising model as characterised by (A10) serves to solve for the large-n limit of the present model. On symmetry grounds we expect that equivalent pairs of sites will have the same λ_{ij} so that knowledge of a periodic Ising model will suffice.

We shall not pursue the general solution further except for the special case of an infinite-ranged interaction,

$$\tilde{J}_{ij} = \hat{J}/N, \qquad \text{all } (ij), \tag{A13}$$

from which results

$$\lambda_{ij} = \hat{\lambda}/N, \qquad \text{all } (ij). \tag{A14}$$

Standard analysis (such as that of Mühlschlegel and Zittartz 1963) then readily gives

$$-kT \ln \tilde{Z}_{\mathrm{eff}} = N\{\hat{J}^{1/2}\hat{\lambda}\mu^2/2 - kT \ln[2\cosh(\hat{J}^{1/2}\hat{\lambda}\mu/kT)]\}, \tag{A15}$$

where

$$\mu = \tanh(\hat{J}^{1/2}\hat{\lambda}\mu/kT). \tag{A16}$$

Similarly

$$\hat{\lambda} = \tilde{J}^{1/2}\mu^2, \tag{A17}$$

so that

$$F = nN\{3\hat{J}\mu^4/4 - kT \ln[2\cosh(\hat{J}\mu^3/kT)]\}, \tag{A18}$$

with

$$\mu = \tanh(\hat{J}\mu^3/kT). \tag{A19}$$

This is identical with the result obtained earlier from the solution to (2.13), as is expected.

The above can be extended to $J^{(1)}$, $h \neq 0$. The analysis is straightforward and we give only the results. (A10) is modified to read

$$\tilde{Z}_{\text{eff}} = \operatorname{Tr} \exp\left[\sum_{(ij)} \left((\tilde{J}_{ij}^{1/2}\lambda_{ij} + J^{(1)})S_iS_j + h\sum_i S_i\right)\Big/ kT\right] \tag{A20}$$

and corresponding modifications are made in the exponents of (A11). For the infinite range problem to be physical we require also $J^{(1)}$ to scale inversely with N:

$$J_{ij}^{(1)} = \hat{J}^{(1)}/N, \qquad \text{all } (ij). \tag{A21}$$

(A18) then becomes

$$F = nN\left(3\hat{J}\frac{\mu^4}{4} + \hat{J}^{(1)}\frac{\mu^2}{2} - kT \ln[\![2\cosh\{[(\hat{J}\mu^2 + \hat{J}^{(1)})\mu + h]/kT\}]\!]\right), \tag{A22}$$

with

$$\mu = \tanh\{[(\hat{J}\mu^2 + \hat{J}^{(1)})\mu + h]/kT\}. \tag{A23}$$

This can also be interpreted as the mean field solution to a finite-range problem if $\hat{J}$, $\hat{J}^{(1)}$ are identified as

$$\hat{J}^{(1)} = \sum_j J_{ij}^{(1)}, \tag{A24}$$

$$\hat{J} = \sum_j J_{ij}^{(2)}. \tag{A25}$$

As before,

$$q \equiv \langle S_i^\alpha S_i^\beta \rangle, \qquad \alpha \neq \beta$$

$$= \mu^2, \tag{A26}$$

and, provided h or J is non-zero,

$$|m| = \langle S_i^\alpha \rangle = \mu; \tag{A27}$$

this last result is in contrast to that for $h = \hat{J}^{(1)} = 0$ for which

$$m = 0, \tag{A28}$$

due to the equivalence of solutions to (A23) with positive and negative μ. Thus any $J^{(1)}$ or h causes any ordered phase to have a ferromagnetic component. It is readily shown from (A23) that the condition for second- to first-order change in the paramagnetic transition temperature for $h = 0$ is in agreement with the prediction of (4.11).

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